Improved Approximation Algorithms for Minimum AND-Circuits Problem via k-Set Cover

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Abstract

Arpe and Manthey [Algorithmica'09] recently studied the minimum AND-circuit problem, which is a circuit minimization problem, and showed some results including approximation algorithms, APX-hardness and fixed parameter tractability of the problem. In this note, we show that algorithms via the k-set cover problem yield improved approximation ratios for the minimum AND-circuit problem with maximum degree three. In particular, we obtain an approximation ratio of 1.199 for the problem with maximum degree three and unbounded multiplicity.

Keywords: approximation algorithms, circuit minimization problem, k-set cover

1. Introduction

The Minimum-AND-Circuit (Min-AC) problem is a circuit minimization problem defined as follows: Given a set of Boolean monomials, find a minimum circuit which consists solely of logical AND-gates with fan-in two and computes these monomials. For example, the monomials may be conjunctive clauses of a DNF formula. A natural way to compute a Boolean function represented by a DNF formula is computing every conjunctive clause, then computing the disjunction of these clauses. According to Charikar et al. [2, Sect. VIII.B], this problem has been studied extensively in the context of automated circuit design, and no approximation guarantees were, however, known. Arpe and Manthey [1] were the first who studied the problem from

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Table 1: Approximation ratios for Min-3-AC

maximum multiplicity	2	3	4	5	$\cdots$	unbounded
Arpe and Manthey [1]	P	1.25	$1.2\overline{6}$	$1.2\overline{7}$		1.278
Our results	-	1.125	1.185	1.198		1.199

a complexity theoretic standpoint, and showed some results including approximation algorithms, APX-hardness and fixed parameter tractability. For details of the results and the background, see their paper [1].

Arpe and Manthey [1] have shown that Min-3-AC, which is Min-AC with instances restricted to monomials of degree at most three, is APX-hard and have given three approximation algorithms called Cover, Match and Greedy. The algorithm Greedy yields an approximation ratio of 1.278. On the other hand, for the case that the degree d is general, the best known approximation ratio is $d - 3/2$, which nearly equals a trivial approximation ratio $d-1$. It remains open if Min-d-AC is approximable with a factor of $o(d)$ or not. In this note, we give improved approximation ratios for Min-3-AC.

Our main idea is as follows. The main part of the algorithm Cover is computing approximately a minimum vertex cover of a three-uniform hypergraph which is obtained from a Min-3-AC instance. The Vertex Cover problem of hypergraphs is essentially equivalent to the Set Cover problem. In this note, we use representation by Set Cover. The *multiplicity* of a set of Boolean monomials is the number of occurrences of each submonomial of size at least two. If the multiplicity of a Min-3-AC instance is at most k, then we can easily show that the cardinality of sets of Set Cover becomes at most k. Such Set Cover problem is called k-Set Cover and can be approximable with a factor of $H_k - 1/2$, where $H_k = \sum_{i=1}^{k} \frac{1}{i}$ [3]. By approximation of k-Set Cover, we improve approximation ratios for the Min-3-AC problem restricted to instances of bounded multiplicity (Sect. 3.1). Moreover, the improvement of approximation ratios for the case that the multiplicity is bounded yields an improved approximation ratio of 1.199 for the Min-3-AC problem with unbounded multiplicity (Sect. 3.2). See Table 1 to compare our results with the previous results.

2. Preliminaries

A *(Boolean) monomial* over a set of Boolean variables $X = \{x_1, \ldots, x_n\}$ is an AND-product of variables of a subset of X. We identify a monomial $M = x_{i_1} \wedge \cdots \wedge x_{i_d}$ with the subset $\{x_{i_1}, \ldots, x_{i_d}\}$, which we denote by M

again. The *degree* of M is $|M|$. We denote by $\binom{M}{2}$ the set $\{S \subseteq X \mid |S| = 2 \wedge S \subseteq M\}$.

An *(AND-)circuit* over X is a directed acyclic graph which consists of n nodes called *input* and the other nodes called *AND-gate*. Each variable $x \in X$ is associated to exactly one of n input nodes. Input nodes have indegree zero and arbitrary outdegree. AND-gate nodes have indegree two and arbitrary outdegree. The size of a circuit C is equal to the number of AND-gates in C. A circuit C *computes* a monomial M if the output of some node in C is equal to M. It computes a set $\mathcal{M}$ of monomials if it computes all monomials in $\mathcal{M}$.

The Minimum-AND-Circuit (Min-AC) problem is defined as follows:

Given a set of monomials $\mathcal{M} = \{M_1, \ldots, M_k\}$ over a set of Boolean input variables $X = \{x_1, \ldots, x_n\}$, find a circuit C of minimum size which computes $\mathcal{M}$.

The total input size of Min-AC is defined as $\sum_{M \in \mathcal{M}} |M|$. In this note, we consider only Min-3-AC, which is Min-AC with instances restricted to monomials of degree at most three.

Let $S \subseteq X$. The *multiplicity* of S in $\mathcal{M}$ is the number of occurrences of S in $\mathcal{M}$ as a submonomial, i.e.,

$$\mathrm{mult}_{\mathcal{M}}(S) = |\{M \in \mathcal{M} \mid S \subseteq M\}|.$$

The *maximum multiplicity* of $\mathcal{M}$ is defined by

$$\mathrm{mult}(\mathcal{M}) = \max_{|S| \geq 2} \mathrm{mult}_{\mathcal{M}}(S).$$

It is equal to the number of occurrences of the most frequent pair of variables in $\mathcal{M}$.

The k-Set Cover problem is defined as follows:

Given a universal set U and a collection C of subsets of U such that each subset in C has a cardinality of at most k, find a minimal subcollection $C' \subseteq C$ whose union is U.

3. Approximation algorithms

3.1. Algorithm for Min-3-AC with bounded multiplicity

In this subsection, we present our approximation algorithm for Min-3-AC with bounded multiplicity via k-Set-Cover and give improved approximation ratios for the case that the maximum multiplicity μ is three or four. In fact, the algorithm gives improved approximation ratios also for the case that

$5 \leq \mu \leq 13$, which is not mentioned below since we give more improved approximation ratios in the next subsection.

For Min-3-AC, without loss of generality, we can assume that all monomials have degree exactly three as Arpe and Manthey [1] have shown. If a given instance includes monomials with degree one or two, then we solve an instance with degree exactly three which is preprocessed as follows, instead of the original instance. Firstly, we delete all monomials with degree one from the instance, since such monomial needs no AND-gate. For each monomial with degree two, we add an AND-gate which computes the monomial. Moreover, if some other monomials with degree three include the monomial with degree two as a submonomial, we add AND-gates which compute the monomials with degree three. Since added AND-gates are necessarily needed gates, we can do such preprocess. Thus we assume that all monomials have degree exactly three. Moreover, we can assume that circuits consist of two layers of AND-gates. The gates of the first layer compute monomials with degree two, and the gates of the second layer output monomials with degree three.

In our algorithm, we first reduce Min-3-AC with maximum multiplicity μ to μ-Set Cover. Let $\mathcal{M}$ be a Min-3-AC instance and $\mathcal{C}$ be a circuit for $\mathcal{M}$. The second layer of $\mathcal{C}$ contains $|\mathcal{M}|$ gates. In the first layer, for each monomial $M \in \mathcal{M}$, one of the pairs contained in $\binom{M}{2}$ has to be computed. Thus, the goal is to find a minimum set of pairs such that each monomial $M \in \mathcal{M}$ has at least one pair. This corresponds to finding a minimum set cover of a Set Cover instance $\mathcal{S}(\mathcal{M})$ obtained as follows. The universal set U equals $\mathcal{M}$. For each pair S contained in $\bigcup_{M \in \mathcal{M}} \binom{M}{2}$, we add the set $\{M \in \mathcal{M} \mid S \in \binom{M}{2}\}$ to a collection C. Since the multiplicity of a Min-3-AC instance is at most μ, (U, C) is a μ-Set Cover instance. Our algorithm $\textsc{Cover}_\mu$ is based on the reduction as above.

Algorithm $\textsc{Cover}_\mu$ for Min-3-AC with maximum multiplicity μ.

1: Input $\mathcal{M} = \{M_1, \ldots, M_k\}$.
2: Compute the μ-Set Cover instance $\mathcal{S}(\mathcal{M})$.
3: Compute a set cover for $\mathcal{S}(\mathcal{M})$.
4: Output the circuit corresponding to the set cover.

The number of gates in the second layer of optimal circuits is k, where k is the number of monomials in $\mathcal{M}$. Let ℓ be the number of gates in the first layer of optimal circuits.

Lemma 3.1. *For $\mu \geq 3$, $\textsc{Cover}_\mu$ outputs a circuit for $\mathcal{M}$ of size at most $k + (H_\mu - \frac{1}{2})\ell$, where $H_\mu = \sum_{i=1}^{\mu} \frac{1}{i}$.*

Proof. μ-Set Cover is approximable with a factor of $H_\mu - \frac{1}{2}$ [3]. The lemma follows from the discussion above. □

Let $\rho_{\textsc{Cover}_\mu}$ be the approximation ratio of $\textsc{Cover}_\mu$. By Lemma 3.1, for $\mu \geq 3$,

$$\rho_{\textsc{Cover}_\mu} \leq \frac{k + (H_\mu - \frac{1}{2})\ell}{k + \ell}, \quad \text{increasing in } \ell.$$

Theorem 3.2. *The* Min-3-AC *problem restricted to instances of maximum multiplicity three and four is approximable with a factor of* 1.125 *and* $45/38 < 1.185$, *respectively.*

Proof. $\textsc{Match}$ achieves the following approximation ratio [1]:

$$\rho_{\textsc{Match}} \leq \frac{\frac{3}{2}k + \frac{1}{2}\ell}{k + \ell}, \quad \text{decreasing in } \ell.$$

For each $\mu \in \{3, 4\}$, we execute $\textsc{Cover}_\mu$ and $\textsc{Match}$, and output the smaller circuit of the two circuits which are output by the two algorithms. The approximation ratio is at most

$$\min\{\rho_1(\ell), \rho_2(\ell)\}, \tag{1}$$

where $\rho_1(\ell) = \frac{k+(H_\mu - \frac{1}{2})\ell}{k+\ell}$ and $\rho_2(\ell) = \frac{\frac{3}{2}k+\frac{1}{2}\ell}{k+\ell}$. We have $\rho_1(\ell) \geq \rho_2(\ell)$ for $\ell \geq \frac{k}{2(H_\mu - 1)}$. Since ρ_1 is monotonically increasing and ρ_2 is monotonically decreasing in ℓ, the minimum in Eq. (1) is attained for $\ell = \frac{1}{2(H_\mu - 1)}k$. It is

$$\rho_1\left(\frac{1}{2(H_\mu - 1)}k\right) = \frac{2(H_\mu - 1) + (H_\mu - \frac{1}{2})}{2(H_\mu - 1) + 1} = \frac{6H_\mu - 5}{4H_\mu - 2}$$

Since $H_3 = 11/6$ and $H_4 = 25/12$, the proof is completed. □

The k-Set Cover problem can be solved in polynomial time if $k = 2$ [3, 4]. Therefore $\textsc{Cover}_2$ gives an alternative proof that the Min-3-AC problem with maximum multiplicity two is in P, which has been proved by Arpe and Manthey [1].

3.2. *Algorithm for* Min-3-AC *with unbounded multiplicity*

In this subsection, we present our approximation algorithm for Min-3-AC with unbounded multiplicity and give an improved approximation ratio of 1.199. The ratio is slightly improved if the maximum multiplicity is at most twelve.

Our algorithm $\textsc{Greedy}_\mu$ for Min-3-AC with unbounded multiplicity is a simple generalization of the algorithm $\textsc{Greedy}$ by Arpe and Manthey [1]. Only step 2 and 8 have been generalized. $\textsc{Greedy}_\mu$ is adjusted with the parameter μ to give the best approximation ratio. $\textsc{Greedy}_2$ is equivalent to $\textsc{Greedy}$. $\textsc{Greedy}_\mu$ greedily adds gates for pairs that occur most frequently in $\mathcal{M}$ until each remaining pair is shared by at most μ monomials. Then it uses the known best algorithm $\textsc{Alg}_\mu$ for Min-3-AC with maximum multiplicity μ.

Algorithm $\textsc{Greedy}_\mu$ for Min-3-AC with unbounded multiplicity.

1: Input $\mathcal{M} = \{M_1, \ldots, M_k\}$.
2: While there exists an $S \in \binom{X}{2}$ such that $|\{M \in \mathcal{M} \mid S \subseteq M\}| \geq \mu + 1$:
3:     Arbitrarily select $S \in \binom{X}{2}$ with maximum $|\{M \in \mathcal{M} \mid S \subseteq M\}|$.
4:     Add a gate computing S to $\mathcal{C}$.
5:     For each $M \in \mathcal{M}$ with $S \subseteq M$:
6:         Add a gate computing M to $\mathcal{C}$.
7:         $\mathcal{M} \leftarrow \mathcal{M} \setminus \{M\}$.
8: $\mathcal{C}' \leftarrow \textsc{Alg}_\mu(\mathcal{M})$.
9: $\mathcal{C} \leftarrow \mathcal{C} \cup \mathcal{C}'$
10: Output $\mathcal{C}$.

Let ρ_μ be the approximation ratio of $\textsc{Alg}_\mu$. As shown in the previous subsection, $\rho_2 = 1$, $\rho_3 \leq 1.125$, and $\rho_4 \leq 45/38 < 1.185$. The number of gates in the second layer of optimal circuits is k, where k is the number of monomials in $\mathcal{M}$. Let ℓ be the number of gates in the first layer of optimal circuits.

Lemma 3.3. *For $\mu \geq 2$, $\textsc{Greedy}_\mu$ outputs a circuit for $\mathcal{M}$ of size at most $max\{\frac{\mu+2}{\mu+1}, \rho_\mu\}k + \rho_\mu \ell$.*

Proof. Let k_1 be the number of monomials in $\mathcal{M}$ that are computed by $\mathcal{C}$ after steps 2-7 and k_2 be the number of monomials in $\mathcal{M}$ that are computed by $\mathcal{C}'$ after step 8. Since each set S selected in step 3 is shared by at least $(\mu + 1)$ monomials, at most $k_1/(\mu + 1)$ gates are added to $\mathcal{C}$ in step 4. In addition, k_1 gates are added to $\mathcal{C}$ in step 6. Let $\mathcal{M}'$ be the set of monomials that remain in $\mathcal{M}$ after steps 2-7. Since $k_2+\ell$ gates are sufficient to compute $\mathcal{M}'$, the number of gates in $\mathcal{C}'$ after step 8 is at most $\rho_\mu(k_2+\ell)$. Since $k_1, k_2 \leq k$,

$$\frac{\mu+2}{\mu+1}k_1 + \rho_\mu(k_2+\ell) \leq \max\left\{\frac{\mu+2}{\mu+1}, \rho_\mu\right\} k + \rho_\mu \ell.$$

□

By Lemma 3.3,

$$\rho_{\text{Greedy}_\mu} \leq \frac{\max\{\frac{\mu+2}{\mu+1}, \rho_\mu\}k + \rho_\mu \ell}{k+\ell}.$$

Greedy$_\mu$ is most balanced and gives the best approximation ratios when $\mu = 4$. (This value may change if ρ_μ gets smaller due to improved approximation algorithms.) By Theorem 3.2, $\rho_4 \leq 45/38$. Thus,

$$\rho_{\text{Greedy}_4} \leq \frac{\frac{6}{5}k + \frac{45}{38}\ell}{k+\ell}, \quad \text{decreasing in } \ell. \tag{2}$$

Since $\ell \geq 0$ and ρ_{Greedy_4} is monotonically decreasing in ℓ,

$$\rho_{\text{Greedy}_4} \leq \frac{\frac{6}{5}k + \frac{45}{38}\ell}{k+\ell} \leq \frac{6}{5} = 1.2.$$

Thus Greedy$_4$ achieves an approximation ratio of 1.2. In the rest, the ratio is slightly improved.

Theorem 3.4. *The* Min-3-AC *problem is approximable with a factor of*

$$\frac{231e^2 - 225}{193e^2 - 190} < 1.199.$$

Proof. Greedy achieves the following approximation ratio [1]:

$$\rho_{\text{Greedy}} \leq \frac{(1+e^{-2})k + 2\ell}{k+\ell}, \quad \text{increasing in } \ell. \tag{3}$$

We execute Greedy$_4$ and Greedy, and output the smaller circuit of the two outputs. By the similar argument of the proof of Theorem 3.2 for Eq. (2) and Eq. (3), the proof is completed. □

Theorem 3.5. *The* Min-3-AC *problem restricted to instances of maximum multiplicity five is approximable with a factor of* $91/76 < 1.198$.

Proof. We execute GREEDY_4. Since maximum multiplicity of instances is bounded to five, each of the ℓ gates at the first layer of optimal circuits can be used for at most five monomials in $\mathcal{M}$. Therefore $\ell \geq k/5$ and GREEDY_4 achieves an approximation ratio at most

$$\max_{\ell \geq \frac{k}{5}} \frac{\frac{6}{5}k + \frac{45}{38}\ell}{k + \ell} = \frac{\frac{6}{5} + \frac{45}{38} \cdot \frac{1}{5}}{1 + \frac{1}{5}} = \frac{91}{76}.$$

□

The similar argument of the proof of Theorem 3.5 yields a little better approximation ratio than the ratio of Theorem 3.4 also for maximum multiplicity μ such that $6 \leq \mu \leq 12$.

4. Concluding Remarks

In Sect. 3.1, we reduce a Min-3-AC instance to a Set Cover instance. All Set Cover instances reduced from Min-3-AC instances have a property that each element is included in at most three sets, since the degree of monomials is at most three. The number of sets which include an element is called *frequency* of the element. In this note, we do not use the property of maximum frequency three. Although the Set Cover problem with maximum frequency three is approximable with a factor of 3 [5], the approximation ratio 3 is also achieved in the algorithm COVER of the previous paper [1] and it does not yield an improved approximation ratio for Min-3-AC. However, we may obtain an improved approximation algorithm if we use both properties of maximum frequency three and k-Set Cover. As far as we know, it is not known whether the k-Set Cover problem with maximum frequency three is better approximable.

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